On the expressivity of total reversible programming languages

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Outline

- Introduction: Motivations
  Primitive Recursive Functions
- Problem: Genesis of SRL
  Questions about SRL
- Solution: Test-For-Zero
  Representation of RPP
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Motivations

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However, reversible computing is relevant for many other applications; as the following classic applications:

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- A wide number of related cases arise when we use a backtracking mechanisms.
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A foundational theory of reversible computing should ease the development of all the above applications but not only.

Summary on Primitive Recursive Functions

Primitive recursive functions (PR) identify a total core of classic computing.

- PR include almost all common (total) functions (on natural numbers).
- PR are simple and endowed with a straightforward semantics.
- PR can be easily extended to grasp the class of all recursive functions.
- PR are sufficient to check if a (finite) computation is correct.

Summary on Primitive Recursive Functions

PR is the smallest class of functions on natural numbers including:

- the zero-function $\mathsf{Z}(x) = 0$,
- the successor $\mathsf{S}(x) := x + 1$
- projections $\pi_i^k(x_1, \ldots, x_k) := x_i$ for all $k \geq i \geq 1$,

and it is closed under:

- composition, viz. the schema that given $g_1, \ldots, g_m, h$ of suitable arities, produces $f(\overrightarrow{x}) := h(g_1(\overrightarrow{x}), \ldots, g_m(\overrightarrow{x}))$, and
- primitive recursion, viz. the function f which is defined from g and h by means of the schema $f(\overrightarrow{x}, 0) := g(\overrightarrow{x})$ and
$$f(\overrightarrow{x}, y + 1) := h(f(\overrightarrow{x}, y), \overrightarrow{x}, y).$$

Summary on Primitive Recursive Functions

Some negative results is known about reversible classes of total functions:

- PR bijections do not include all total computable reversible functions.
- PR bijections are not closed under inversion.
- The class of all computable bijections cannot be recursively enumerated.

SRL Genesis

- In 1968 Dennis Ritchie in his doctoral thesis "Program Structure and Computational Complexity" proposed the LOOP language (an old-fashion FOR language). LOOP is complete w.r.t. to primitive recursive functions.

- In this paper we focus our attention on SRL and its variants, namely a family of total reversible programming languages introduced in 2003 by Armando Matos conceived as a restriction of LOOP.

- The main difference between SRL languages and LOOP languages is that their registers store (positive and negative) integers.

$$\mathtt{P} \ ::= \mathsf{inc}\,\varkappa \mid \mathsf{dec}\,\varkappa \mid \underbrace{\mathsf{for}\,\varkappa\,(P)}_{\text{in SRL: } \varkappa \notin P} \mid P; P$$

Questions about SRL

Many questions have been posed about the expressivity of SRL.

1. Is the program equivalence of SRL decidable?
2. Is it decidable if a program of SRL behaves as the identity?
3. Is decidable whether a given program is an inverse of a second one?
4. Is SRL primitive-recursive complete?
5. Is SRL sufficiently expressive to represent RPP (or RPRF)?

In this work we answer to all them, by showing that:

> "a choice-operator can be implemented in SRL."

Test-for-Zero

- A Truth Values is represented by two-ordered registers r_t, r_f:
  - **true** is represented by $r_t, r_f \leftarrow 1, 0$;
  - **false** is represented by $r_t, r_t \leftarrow 0, 1$.

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- $\texttt{if}\ (r_t == 1 \wedge r_f == 0)\ \texttt{then}\ P_0\ \texttt{else}\ P_1$
  can be simulated by for $r_t\ (P_0)$; for $r_f\ (P_1)$

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- Therefore, we need a test-for-zero.
  If R is a register and r_t, r_f form a truth-pair initialized to $0, 1$,
  then we are looking for an operator such that:
  - if $R \neq 0$ then all registers are unchanged after the test;
  - if $R = 0$ then all registers are unchanged, but r_t, r_f which are swapped.

Test-for-Zero

- To decide the parity is easy:

$$\begin{vmatrix} n \\ 1 \\ 0 \\ 0 \end{vmatrix} \ \mathsf{for}\, r_0(\mathsf{swap}(r_1, r_2); \mathsf{for}\, r_1(\mathsf{inc}\, r_3)); \ \begin{vmatrix} n \\ b_{\mathit{even}} \\ b_{\mathit{odd}} \\ n^{\bullet}/2 \end{vmatrix}$$

- The *Fundamental Theorem of Arithmetic* :

  each $n \neq 0$ admits a unique decomposition (up to the order of its factors)

$$(\pm 1)2^k p_1 p_2 \cdots p_m$$

  where $k \geq 0$ and, each p_i is a (positive) odd-prime number.

Test-for-Zero

Procedure `isLessThanOne`

Let r_2, r_3 and r_5, r_6 be truth-pairs initialized to true and let r_4 be a zero-ancilla. Let both r_0 and r_1 contain the value N. Then:

$$\mathsf{for}\, r_0 \left(\begin{array}{ll} \mathsf{for}\, r_5(\mathsf{for}\, r_1(\ \mathsf{swap}(r_2, r_3);\ \mathsf{for}\, r_2(\mathsf{inc}\, r_4)\ ));\quad & \texttt{/* SP0 */} \\ \mathsf{for}\, r_3(\mathsf{swap}(r_5, r_6)); & \texttt{/* SP1 */} \\ \mathsf{for}\, r_5(\ \mathsf{for}\, r_4(\mathsf{dec}\, r_1);\ \mathsf{for}\, r_1(\mathsf{dec}\, r_4)\ ); & \texttt{/* SP2 */} \\ \mathsf{for}\, r_6 \left(\begin{array}{l} \mathsf{for}\, r_1(\ \mathsf{for}\, r_2(\mathsf{dec}\, r_4);\ \mathsf{swap}(r_2, r_3)\ ); \\ \mathsf{for}\, r_1(\mathsf{inc}\, r_4);\ \mathsf{for}\, r_4(\mathsf{inc}\, r_1) \end{array} \right) & \texttt{/* SP3 */} \end{array} \right)$$

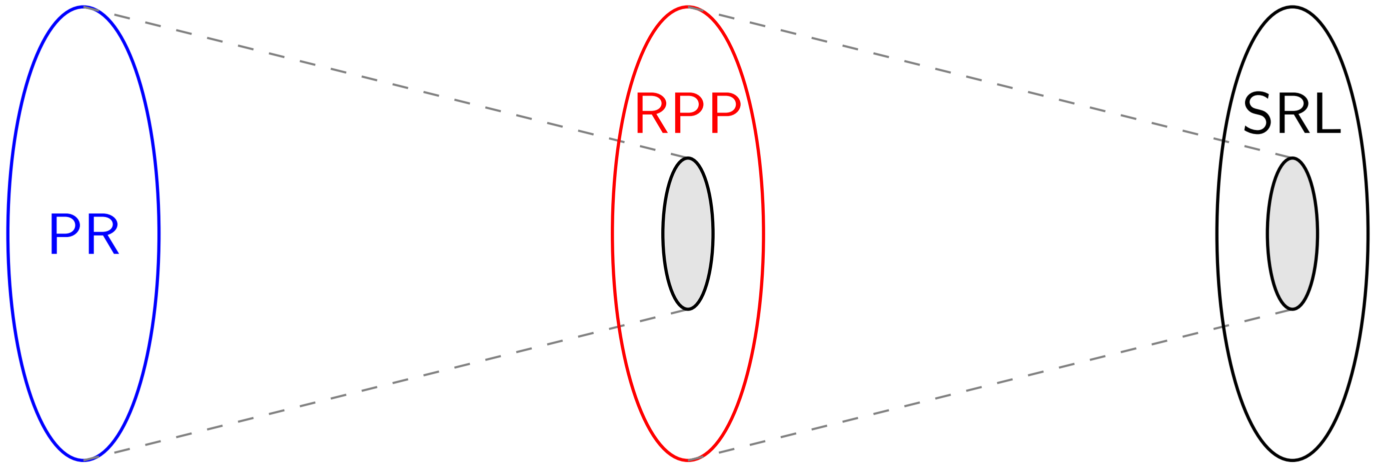
leaves true in the truth-pair r_5, r_6 if and only if N is strictly lower than 1.

RPP is a sub-class of endofunctions on $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

1. RPP^1 includes successor, predecessor negation
2. RPP^2 includes the swap
3. If $f, g \in \mathsf{RPP}^k$ then, RPP^k includes their series-composition
4. If $f \in \mathsf{RPP}^j$ and $g \in \mathsf{RPP}^k$, then RPP^{j+k} includes their parallel composition
5. If $f \in \mathsf{RPP}^k$, then the finite iteration $\mathsf{It}\,[f]$ belongs to RPP^{k+1}
6. Let $f, g, h \in \mathsf{RPP}^k$. The selection $\mathsf{If}\,[f, g, h]$ belongs to RPP^{k+1} and it is the function defined as:

$$\mathsf{If}\,[f, g, h]\,(\langle x_1, \ldots, x_k, z\rangle) := \begin{cases} (f \parallel \mathsf{Id})\,(\langle x_1, \ldots, x_k, z\rangle) & \text{if } z > 0\ , \\ (g \parallel \mathsf{Id})\,(\langle x_1, \ldots, x_k, z\rangle) & \text{if } z = 0\ , \\ (h \parallel \mathsf{Id})\,(\langle x_1, \ldots, x_k, z\rangle) & \text{if } z < 0\ . \end{cases}$$

Conclusions



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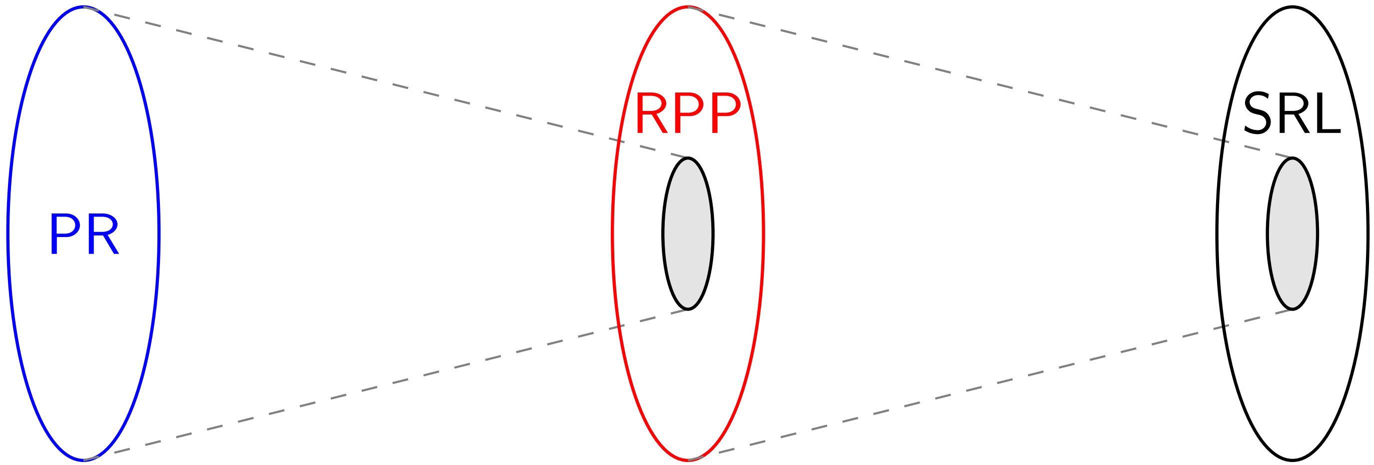

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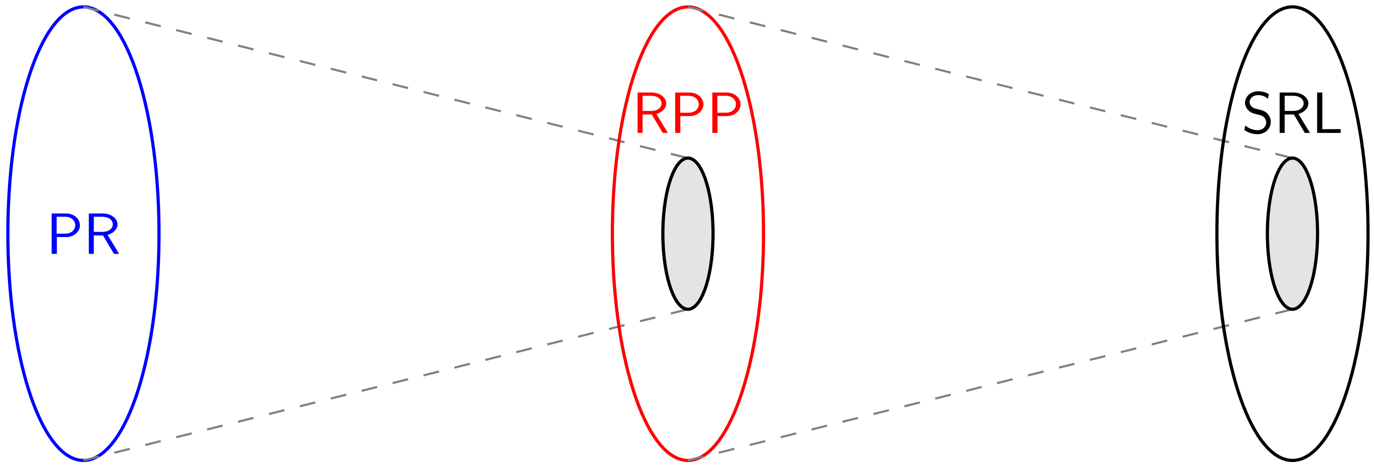

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Future Works

- ☐ Turing-complete extensions
- ☐ Kleene's theorems, ...
- ☐ Complexity Hierarchies, ...

End ...

thank you!